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Reputation in Multi-unit Ascending Auction with Common Values

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Abstract

This paper considers a multi-unit ascending auction with two players and common values. A large set of equilibria in this model is not robust to a small reputational perturbation. In particular, if there is a positive probability that there is a type who always demands many units, regardless of price, then the model has a unique equilibrium payoff profile. If this uncertainty is only on one side, then the player who is known to be normal lowers her demand in order to stop the auction immediately at the reserve price. Hence, her possibly committed opponent buys all the units she demands at the lowest possible price. If the reputation is on both sides, then a War of Attrition emerges.

Keywords: Multi-unit auction, uniform price, ascending auction, reputation, aggressive bidding.

JEL Classification: D44

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1 Introduction

In the simultaneous ascending auction, the bidding continues until the total demand decreases so that it matches the supply. Despite great progress in understanding various incentives faced by players in this auction, the simplest theoretical model of such an auction with many units for sale and common values suffers from a serious deficiency: there is a multiplicity of equilibria, and consequently the model has virtually no predictive power.

This paper shows that this multiplicity of equilibria is not robust. Introduce a positive probability that players may be of a behavioral type that demands large quantities, even if prices are high. This perturbation creates incentives for normal players to mimic those who are programmed to demand large quantities. Then there is a unique equilibrium payoff profile and that fact does not depend on the size of the perturbation. This result is obtained in a model of ascending auction with two players and multiple units of a homogeneous good for sale, in which the value of a unit is the same for both players regardless of quantities acquired, but it is not necessarily common knowledge.

If the reputation is one-sided (only one player may be behavioral, but the other one is certain to be normal), then in any equilibrium the auction stops immediately. The player who has no reputation decreases his demand to the level that clears the market, in order to prevent the price from rising. If the reputation is two-sided, then there is delay.

Riedel and Wolfstetter (2006) show that if the marginal values are known, decreasing in quantity and not equal across agents, then there is a unique equilibrium through iterated elimination of dominated strategies, in which the division of objects is efficient and the price is equal to the reserve price and therefore the minimal revenue. Their case does not cover common values and thus the analysis below can be viewed as a complement of their work.

The central parameter for an assessment of the equilibrium of an ascending auction with common values is not what is known about the values of the object for sale, but how symmetric/asymmetric the reputational profile is. When reputation across players is asymmetric – then the seller will not get any revenue beyond the reserve price, *even if the normal players have private information* about the value of the object. On the other hand, when reputation is symmetric, then the revenue should be higher than the reserve price, *even if the normal players do not have private information* about the value of the object. The detailed discussion of these results is presented after the model.

2 Literature

It is known that there is a great multiplicity of equilibria in a standard ascending or uniform price multi-unit auction, particularly if the values are common. Very low prices, even equal to an exogenous reserve price, may be supported by an equilibrium. For instance, assume that the equilibrium implicitly splits the total number of units among players who keep demanding their designated shares for any price. No player has any incentive to deviate from this strategy profile. Any deviation to a lower demand does not affect the price, but assigns fewer units to the player and hence results in lower payoff. Any deviation to a higher demand rises the price but ultimately does not affect the assignment, again resulting in lower payoff. Equilibrium strategy profile results in obtaining the agreed share at the lowest possible price. Other allocations and prices can be supported as equilibrium outcomes too.

Obviously, this collusive-seeming equilibrium is not revenue-maximizing. A possibility of low revenues was first noticed by Wilson (1979), in the context of a divisible-object and sealed-bid version of the model, see also Back and Zender (1993). Riedel and Wolfstetter (2006) has already been mentioned.

Some recommendations were given to the seller who, for some reasons, must use uniform price mechanism. Back and Zender (1999) and McAdams (2007) give the seller an opportunity to adjust the supply strategically after players announced their demands. Since players' collusive behavior is sensitive to this quantity, they are not able to coordinate so well.

Another approach assumes that there are some noisy players who purchase some part of the total supply, leaving an uncertain residual for the strategic players. The probability of such a supply variation can be arbitrarily small, and still a unique equilibrium is selected as long as this noise has a full support. This method of equilibrium selection was introduced by Klemperer and Meyer (1989); Rostek et al. (2009) is a recent application. The model in this paper introduces a perturbation too, but in a different context and with different implications.

All of the above papers consider multi-unit auctions in a sealed-bid context. In contrast to that, this project focuses on ascending version of the model, similar to Milgrom (2000) or Cramton (2006). This rises a possibility of interesting dynamics, such as collusion (for instance, Cramton and Schwartz (2002)). This paper studies dynamic reputational effects. There is a small literature on auctions and reputations in the repeated auction framework (Bikhchandani (1988), Kwiek (2011)).

From the point of view of the reputation literature the model of this paper is closely related to Myerson's (1991, section 8.8) model of bargaining with reputation. A multi-unit ascending auction can be seen as a mechanism in which players negotiate/bargain how to split the objects among themselves. One difference is how the cost of bargaining is specified – in the classical bargaining model this is discounting, in the multi-unit ascending auction this is through increasing price. The connection between bargaining and ascending multi-unit auctions was made by Ausubel and Schwartz (1999)

The most important contribution of this study is an extension of Myerson's (1991) result to the case in which common values are not commonly known, in the context of an auction. In particular, reputational effects completely dominate what players may know about the value of the objects. For instance, suppose that player one knows the value and player two does not; standard logic of the winner's curse suggests that player two should yield early. However, if player two has a reputation as defined below (no matter how small) and player one does not, it is player one who would yield immediately.

Myatt (2005) studies a War of Attrition with ex ante asymmetry in values rather than with common values; his result is that the player with lower private value exits almost immediately if the two-sided reputational perturbation is small enough. To complement this result, this paper provides an example with one-sided reputation, like Myerson (1991), and demonstrates that there may be multiplicity of equilibria if values are ex ante asymmetric, as in Myatt (2005).

3 Model

Players, types and payoffs. There are $L \geq 3$ units of a good for sale. Let $r \geq 0$ be an exogenous reserve price. There are 2 players, who can buy at most $L - 1$ units.[1]

The information is asymmetric in two ways. At the beginning of the auction, nature chooses for each player $i = 1, 2$ (i) a type $t_i \in \{0, x_i\}$, where $t_i = 0$ denotes a "normal" type and $t_i = x_i$ denotes a "behavioral" type; and (ii) a signal $\theta_i \in \Theta_i$ about the value of the good. For simplicity assume that the set of signal profiles $\Theta = \Theta_1 \times \Theta_2$ is finite; let $\theta = (\theta_1, \theta_2)$. Bidder i learns his type and his signal, (t_i, θ_i). Assume that θ, t_1 and t_2 are independent, but θ comes from an unrestricted joint probability distribution, where q_θ is the probability of θ.

With probability $1 - \mu_i$ player i is a normal type, $t_i = 0$. The gross value of one unit when θ is realized is $v_\theta > 0$. This formulation makes sure that the value of the unit is constant across agents and does not depend on the quantity acquired by the player. This value however does depend on the profile of signals, and therefore is only partially known to player i, who knows the realization of signal θ_i but not of θ_j. A normal type of player i has the realized net payoff equal to $(v_\theta - p) z_i$, where p is the final price of each unit and z_i is the final allocation of units to player i.

With probability $\mu_i \geq 0$ player i is a behavioral type, $t_i = x_i$. The behavioral type always demands x_i units for any price, a quantity that is treated as exogenous in this paper. To assure that seller's revenue is bounded, assume that the behavioral type reduces his demand from x_i to zero at price $\max_\theta v_\theta$.

[1] Assuming that bidders may demand all L units introduces more cases to consider without adding any insights.

Assumptions about the dynamic structure. The format of an auction is ascending. The extensive game form will clarify what is meant by this statement, but will reflect the following assumptions. The price is continuously rising and each player publicly announces how many units he wants to buy at a current price. Bidders announce their individual demands by means of a dial that can be rotated only towards lower values. They can decrease their demand only by one unit at a time, but they can react instantaneously to such demand reductions. If two players decide to decrease demand at the same price, nature uses a fair coin to determine whose reduction is announced. The auction stops at the market clearing price, the first price where the excess demand is zero.

The extensive game form. After the players learn their types and signals, their first move is to announce simultaneously their starting demands at the initial reserve price. If the initial excess demand, denoted $\bar{n}$, is non-positive then the auction stops and the normal types receive payoffs implied by these demands and the reserve price. Otherwise, if the initial excess demand is $\bar{n} > 0$, the game enters the ascending price phase. This multi-unit ascending auction will be regarded as a series of simple timing games.

Formally, define $n \in \{\bar{n}, \bar{n} - 1, ..., 1\}$ to be a stage of the game in which the excess demand is n. In other words, stages of the ascending auction are counted backwards, with stage 1 being the last one. If one lets z_i^n to be the demand of player i in stage n, then obviously $L + n = z_1^n + z_2^n$.

In each stage n, player $i = 1, 2$ chooses a price, b_i, at which she intends to reduce her demand by one unit. This price will also be referred to as her "bid". This bid cannot be lower than a reserve price in this stage, denoted p^n; the initial reserve price is exogenously given, $p^{\bar{n}} = r$. If player 1 chooses a bid lower than player 2, then her bid defines a new reserve price for the next stage, $p^{n-1} = b_1$. The new demand of player 1 is $z_1^{n-1} = z_1^n - 1$, and the demand of player 2 remains unchanged at $z_2^{n-1} = z_2^n$. If there is a tie, then the player whose demand is to be reduced is determined randomly. The other player, who also intended to reduce her demand at this price, may reduce her demand at the same price but at the next stage.

The public history at the beginning of stage n consists of prices at which previous reductions of demand occurred and the resulting demands. Thus, the history at the beginning of stage n is

$$h^n = \left\{ \left(p^{\bar{n}}, z_1^{\bar{n}}, z_2^{\bar{n}}\right), \left(p^{\bar{n}-1}, z_1^{\bar{n}-1}, z_2^{\bar{n}-1}\right), ..., \left(p^n, z_1^n, z_2^n\right) \right\}$$

Elements of h^n are related in the obvious way,[2] following from the auction procedure described above. The final element of this list must have $n > 0$ for the auction to still continue after price p^n. The new history in the next stage is $h^{n-1} = \left\{ h^n, \left(p^{n-1}, z_1^{n-1}, z_2^{n-1}\right) \right\}$. Let the probability of a normal type of player i at the beginning of stage n be μ_i^n, where the initial probability is an exogenous parameter of the game, $\mu_i^{\bar{n}} = \mu_i$.

Strategy. Let $F_i\left(b_i|\theta_i\right)$ be a probability that a normal player i who observed θ_i decreases her demand at or before price $b_i \geq p^n$, where the dependence on the history h^n is implicit. This can also be expressed as $\Pi_i\left(b_i|\theta_i\right)$, the probability that (unconditional) player i decreases her demand at or before b_i. A strategy for the entire game is a collection of these functions, one for each history, and a probability distribution over $\{0, 1, ..., L-1\}$ which picks the initial demand.

4 Behavioral type on one side

This section assumes that player i is known to be normal, $\mu_i = 0$, but the probability that player j is of behavioral type is $\mu_j > 0$. Consider strategies in which normal player j starts demanding x_j at the reserve price and suppose that at the beginning of this ascending auction there is some excess demand, $\bar{n} \geq 1$.

The following is the main result. The proof is in the Appendix.

[2] In particular, $p^{\bar{n}} = r$ is the initial price; $z_i^{\bar{n}} \leq L-1$ is the initial demand of bidder i. The price at which one of the players reduced her demand is p^η, and the demand of player i is z_i^η, where $\eta = \bar{n}-1, ..., n$ is the excess demand resulting from this reduction. The price at which demand was reduced becomes the new "reserve" price for the next stage, $p^\eta \geq p^{\eta+1}$. If $p^\eta = p^{\eta+1}$, then after one of the bidders decreased her demand at price $p^{\eta+1}$, one of the bidders decreased the demand again at the same price. Demand reductions occur only by one unit, $z_i^\eta \in \left\{ z_i^{\eta+1} - 1, z_i^{\eta+1} \right\}$ for both $i = 1, 2$ and $z_i^\eta = z_i^{\eta+1} - 1$ for some i.

Proposition 1 *Consider any stage $n \geq 1$ with initial price p^n, in which $\mu_j^n > 0$, $\mu_i^n = 0$. If player j has a signal such that her conditional expected value exceeds this initial price, then the only equilibrium market clearing price is p^n and the allocation to player j is x_j units.*

This is closely related to the result developed by Myerson (1991, section 8.8) in the context of bargaining. The proof measures two forces that would have to occur in any equilibrium in which both players do not reduce their demands immediately. The first force is that after player j reduces his demand (thus revealing his normality) player i must obtain a particularly high payoff, call it w_i. This is because if j does not reduce his demand (which happens with positive probability), then player i obtains a low payoff from insisting on her high initial demand for some time and yielding at a higher price. Since the expected payoff of player i over these two events has to be at least as good as the available option of reducing the demand immediately, the promise of w_i must be attractive. The second force is that when player j reduces her demand, she must obtain a high payoff, call it w_j, at least as high as the available option of waiting until player i ultimately reduces her demand. The proof shows that these requirements for high payoffs w_i and w_j are not simultaneously feasible.

Proposition 1 makes sure that the construction of Myerson (1991) works in the current auction context and even if common values are not commonly known. Some remarks about common values should be made.

The first interesting observation is that Proposition 1 does not depend on set Θ or the distribution of the signals. This may strike as surprising. For example, suppose that player i knows the true value of a unit, while player j does not. One may argue that this additional knowledge gives player i an advantage: it makes bidding less attractive to player j, due to winner's curse. This may force player j to yield before player i. This argument was formalized by Milgrom and Weber (1982, Theorem 7) in the context of common-value single-unit auctions.[3] Yet, Proposition 1 states that this consideration

[3] They showed that if the signal of one of the players is a garbling of the other signal, then one of the players wins immediately with probability one, although it does not have to be the player with the superior information.

is completely dominated by reputation. The literature on single-unit auctions with almost common values (Bikhchandani (1988), Klemperer (1998)) observes that a slight perturbation of the common-value model leads to a result that is apparently very similar to Proposition 1: a small ex ante asymmetry between players leaves only one equilibrium, which is very asymmetric. However, in a certain sense this similarity is deceiving, for winner's curse does not play any role in Proposition 1. For example, if one assumes that common value is commonly known, thus eliminating winner's curse, then Milgrom and Weber (1982), Bikhchandani (1988) and Klemperer (1998) all cease to exhibit strong asymmetries. In contrast to this, Proposition 1 is valid even if Θ is a singleton. It really is the force identified by Myerson (1991) that drives Proposition 1.

This rises a question: if winner's curse is not the driving force behind Proposition 1, can the assumption about common values be relaxed? It turns out that the answer is no. The following example considers players whose values are asymmetric ex ante, a case studied by Myatt (2005), but focuses on one-sided reputation, like in Myerson (1991). In contrast to Proposition 1, there may be a multiplicity of equilibria. More importantly, if reputation was two-sided, then the outcome would be determined completely by the asymmetry of values, and completely independent on the asymmetry of reputations, in the limit. In other words there is no counterpart of Proposition 1, and the relevant environment is Riedel and Wolfstetter (2006).

Example 1 *Suppose that* $r = 0$, $x_i = x_j = 2$, *and* $L = 3$. *Values are commonly known but they are not common. In particular, assume that* i *is value-strong, but reputation-weak. That is, let* $v_j < v_i$ *and assume that player* i *is known to be normal,* $\mu_i = 0$ *and player* j *is possibly committed* $0 < \mu_j \leq 1 - v_j/v_i$. *For any number* a_i *such that* $0 \leq a_i \leq 1/v_j$, *a pair of functions*

$$\begin{aligned}\Pi_i(b) &= a_i b + (1 - a_i v_j) \\ \Pi_j(b) &= b/v_i\end{aligned}$$

for $0 \leq b \leq v_j$ *is an equilibrium. Similarly, for any number* $a_j \geq \mu_j / (v_i - v_j)$, *a pair of functions*

$$\begin{aligned}\Pi_i(b) &= b/v_j \\ \Pi_j(b) &= a_j b + (1 - a_j v_i)\end{aligned}$$

for $0 \leq b \leq v_j$ *is an equilibrium.*

If $a_j = \mu_j / (v_i - v_j)$, *then the equilibrium is the limit point of the sequence of unique equilibria in the two-sided reputation model, when one considers a sequence of models with* $\mu_i \to 0$. *Notice that if we further assume that* $\mu_j \to 0$, *we have that* $\Pi_j(b) \to 1$ *for all* b *and hence the value-strong player wins the auction at the lowest price with probability one, regardless of what the relative reputational profile is.*

5 Some implications

What if reputation is on both sides, $\mu_j > 0$ and $\mu_i > 0$? The equilibrium would resemble a War of Attrition, in which normal types of both players mimic their behavioral types for a certain period of time, until one of the players reduces her demand, revealing that she is normal. This action induces a one-sided reputational phase, as described above. In other words, as soon as one of the players reduces her demand at a certain price, she keeps reducing it at this price until total quantity demanded reaches quantity supplied.

The probability is strictly less than one that market clearing occurs at the reserve price and, consequently, the seller's expected revenue is higher than the reserve price. To see this, suppose that this is not true and in fact there is an equilibrium in which the normal type of a player, say i, reduces her demand at the reserve price with probability one. Her opponent, player j, who observes that i does not reduce demand must infer that she faces a behavioral type and hence must respond by reducing her demand immediately. But then, normal type of player i could mimic her behavioral type and receive a payoff arbitrarily close to her maximum, proving that this is not an equilibrium.[4]

[4] If values are common and commonly known (Θ is a singleton), then the equilibrium

As far as the revenue is concerned, the results for the common value case studied above are different than for the private value context studied in the literature. Riedel and Wolfstetter (2006) work with the assumption that commonly known valuation functions of two players cross once, and thus values are not common. Their game is dominance solvable and the efficient division of units across players occurs at the reserve price. If one considers a situation in which players' valuation is their private information, then one can conjecture that the expected price in equilibrium would be necessarily higher than the reserve price.

The situation is somewhat different in the case of the common-value auctions studied in this paper. Predictions depend on how symmetric the reputation profile is, and not on the quality of signals that players may have about the objects' value. If the reputation profile is two-sided, and in particular if it is symmetric, then the seller will obtain a revenue beyond the reserve price *even if* players do not have any private information about the objects' value (as in footnote 4). On the other hand, if reputations are very asymmetric, then the revenue of the seller will be close to the reserve price, *even if* players do have private information about the value of the objects for sale (as in Proposition 1).

The interpretation of the analysis presented in this paper depends crucially on whether the existence of behavioral types is understood to be significant (either μ_i or μ_j are far from zero), or whether it is a near impossibility (both μ_i and μ_j are close to zero).

One can encounter situations in which the first of these cases occurs. One possible justification of behavioral types is worth mentioning here. Imagine a type of a player for whom various objects are complements. For instance, telecommunication firms may face very strong geographical complementari-

can be derived explicitly along the lines of Abreu and Gul (2000). The unique perfect Bayesian equilibrium payoff profile is generated by a strategy profile that involves

$$\begin{cases} \Pi_j(b) = 1 - \left(\frac{v-b}{v-r}\right)^{\omega_i} \\ \Pi_i(b) = 1 - \mu_i \left(\mu_j\right)^{(\omega_j/\omega_i)} \left(\frac{v-b}{v-r}\right)^{\omega_j} \end{cases}$$

for $b \in \left[r, v-(v-r)\left(\mu_j\right)^{(1/\omega_i)}\right]$, where $\omega_i = y_i/\left(x_i - y_i\right)$ and $0 < \left(\mu_i\right)^{\omega_i} \le \left(\mu_j\right)^{\omega_j}$.

ties, whereby a license in one area is worth a lot only if a license in a neighboring area is acquired too. In the context of the above model, assume that a bundle of x units is worth vx to this type, if all x units are acquired, and zero otherwise. Such a type has a weakly dominant strategy to stay in the auction until the price reaches v, thus behaving exactly like a behavioral type assumed in this paper. If indeed behavioral types are justified, this paper provides a useful model to judge equilibrium behavior. The equilibrium is robust, in the sense that a small change of the parameters of the model leads to a small change of the equilibrium.

If behavioral types are unrealistic ex ante, then the interpretation of the analysis presented in this paper leads to rather ambiguous conclusions for an external observer (such as an econometrician or the seller). It is difficult to observe if and in which way the reputational profile is asymmetric. In this case, one should not have much confidence in the outcome of the auction. In other words, all equilibria of the unperturbed model should be treated seriously, because any of them can be approximated by a unique equilibrium of the perturbed model.

It is worth mentioning that there are cases that are inherently symmetric, such as when players meet for the first time, or if they are anonymous.

The model presented above is simple, so it is natural to ask how robust it is to different assumptions. One possible objection is to the assumption of rigidity of the behavior of the behavioral type: it always demands an exogenous number x_i. This assumption is made for convenience and can be relaxed. For instance, one could consider more complicated behavioral types who reduce demand with some density as the price goes up. However, as long as the probability is nonzero that they fail to reduce demand, Proposition 1 applies.

One can also consider a more complicated game, in which at the beginning of the auction there is private information about the size of x_i. When choosing their initial demand, normal types choose which behavioral type they want to mimic. After this initial demand is announced, normal players believe that they may face either a normal type or one particular behavioral type who sticks to the observed level of demand. From now on the game path follows a two-sided War of Attrition. Obviously, the equilibrium behavior and seller's revenue depend on the ex ante distribution of behavioral types, but the complication of having more behavioral types does not contribute to our understanding of the observations made above.

6 Appendix. Proof of Proposition 1

The following lemma provides an inductive step (where $E_{\theta_i|\theta_j,h^n}v_\theta$ is the expected value conditional on θ_j and history h^n).

Lemma 1 *Fix $n = 0, 1, ..., 2(L-1) - 1$. Suppose that in any stage n in which the reputational profile is $\mu_j^n > 0$, $\mu_i^n = 0$ and the signal θ_j is such that $p^n < E_{\theta_i|\theta_j,h^n}v_\theta$, it is true that the only equilibrium market clearing price is p^n and the equilibrium allocation to player j is x_j units.*

Then in any stage $n+1$ in which the reputational profile is $\mu_j^{n+1} > 0$, $\mu_i^{n+1} = 0$ and the signal θ_j is such that $p^{n+1} < E_{\theta_i|\theta_j,h^{n+1}}v_\theta$, it is true that the only equilibrium market clearing price is p^{n+1} and the equilibrium allocation to player j is x_j units.

Before the proof of this lemma is provided, a simpler version of this result is presented. In particular, assume that Θ is a singleton, so that the value of the object is commonly known. Assume also that the excess demand is one unit, so that $n = 0$ and the inductive hypothesis is satisfied by construction.

Ignore subscript in x_j and let $y = L - x$ be a residual quantity for player i, once the behavioral demand of player j is fully satisfied. Let $\Pi_j(b)$ be the probability that player j decreases his demand for the first time before or at b. Obviously, $\Pi_j(b) \leq 1 - \mu_j^1 < 1$ for all b. Define R_j to be the greatest price at which player j reduces his demand. Notice that $R_i \leq R_j$ and $\Pi_i(R_i) = 1$. This is because at any price higher than R_j player i believes with probability one that j is not going to reduce his demand.

Write R for R_i and contrary to the claim of the Lemma, suppose that R is strictly greater than the reserve price, $p^1 < R$.

For any $b \in (p^1, R)$, player j could reduce demand either at prices in the interval $[b, R]$ or at prices higher than R (including never reducing demand).

Let ζ^b be the probability that player j reduces his demand at prices higher than R, conditional on reaching price b. That is

$$\zeta^b = \frac{1 - \Pi_j(R)}{1 - \Pi_j(b)}$$

Let $\left(w_i^b, w_j^b\right)$ be the expected payoff profile at b conditional on player j reducing his demand at prices less than R. Beyond its feasibility, one does not know anything specific about this payoff profile, because if j reduces her demand prior to R, she reveals her normality and the game may realize multiple equilibrium payoff profiles. The core of the proof lies in points 3, 4 and 5 below. Point 3 states the feasibility constraint on $w_i^b + w_j^b$. Point 4 states that the promise w_i^b has to be high enough for i to follow this equilibrium. Point 5 states that the promise w_j^b has to be high enough for j to follow the equilibrium. The rest of the proof demonstrates the contradiction between these feasibility and incentive requirements.

1. It must be that $R < v$. Contrary to the statement, assume $R = v$. Player i compares the payoff from reducing demand at price b and at a later price $b^{\#} \in (b, R)$. The minimal payoff that this player gets from reducing at b is $(v - b)\, y$. The maximal payoff if i reduces at $b^{\#}$ is no higher than

$$(v - b)\, x \left(\Pi_j\left(b^{\#}\right) - \Pi_j(b)\right) + \left(1 - \Pi_j\left(b^{\#}\right)\right)\left(v - b^{\#}\right) y$$

which assumes that if j reduces at a price in the interval $\left[b, b^{\#}\right)$, then the quantity x demanded by player i is fully satisfied at the lowest possible price b. The former cannot exceed the latter, which leads to an inequality

$$\left(1 - \Pi_j\left(b^{\#}\right)\right)\left(v - b^{\#}\right) y \geq (v - b)\left(y - x\left(\Pi_j\left(b^{\#}\right) - \Pi_j(b)\right)\right)$$

Consider the limit of this inequality as $b^{\#} \to R$. The left-hand side converges to zero, since $R = v$. If b is close enough to R we obtain a contradiction in the form of $\lim_{b^{\#} \to R} \Pi_j\left(b^{\#}\right) - \Pi_j(b) \geq y/x$.

2. Probability ζ^b converges to one as $b \to R$. Contrary to this statement, assume that there is a positive probability mass that player j reduces her demand at price R. Then player i who is supposed to reduce demand at a price close to R, would benefit from delaying this reduction to a price just after R. The cost of this delay can be made arbitrarily small, but the benefit is bounded away from zero by point 1.

3. Total payoff at price b, conditional on player j reducing his demand at some price higher than b, cannot exceed the total surplus available at price b. Namely,
$$w_i^b + w_j^b \leq (v - b)(x + y)$$

4. The expected payoff of i from his high demand has to be at least as high as the expected payoff from reducing the demand immediately to y at a current price b:
$$\zeta^b (v - R) y + \left(1 - \zeta^b\right) w_i^b \geq (v - b) y$$
Replace w_i^b by its upper bound implied by step 3,
$$-\zeta^b (R - b) y + \left(1 - \zeta^b\right)(v - b) x - \left(1 - \zeta^b\right) w_j^b \geq 0 \tag{1}$$

5. With positive probability player j reduces his demand at prices below R. The payoff from this cannot be lower than the lowest payoff from maintaining the demand for x units. Namely,
$$w_j^b \geq (v - R) x \tag{2}$$

6. Replace w_j^b in inequality (1) with its lower bound in inequality (2) to conclude that ζ^b is bounded away from 1 by a term involving only x and y
$$\zeta^b \leq \frac{x}{x + y} < 1$$

This contradicts step 2. Thus $R = p^1$.

The main difference between this simplified version and the real proof of the lemma is that the latter has to take into account the fact that there are privately known payoff-relevant signals. The difficulty comes from the fact that some inequalities are valid signal-by-signal, while some are valid only as conditional expectations for different conditions.

Proof. Consider excess demand of $n+1$ units at the starting price p^{n+1}. Fix an equilibrium.

Let $F_j(b|\theta_j)$ be the probability that normal player j decreases his demand for the first time before or at b, conditional on observing signal θ_j; let $\Pi_j(b|\theta_j) = \left(1-\mu_j^{n+1}\right) F_j(b|\theta_j)$. Define $\tilde{R}_j(\theta_j)$ to be the greatest price at which player j of signal θ_j reduces his demand

$$\tilde{R}_j(\theta_j) = \min\left\{b : \Pi_j(b|\theta_j) = \lim_{b' \to \infty} \Pi_j(b'|\theta_j)\right\}$$

Let $R_j = \max_{\theta_j} \tilde{R}_j(\theta_j)$. Similarly define $\Pi_i(b|\theta_i)$, $\tilde{R}_i(\theta_i)$ and R_i. Obviously, $\Pi_j(b|\theta_j) \leq 1-\mu_j^{n+1} < 1$ for all b and θ_j.

Notice that $R_i \leq R_j$ and $\Pi_i(R_i|\theta_i) = 1$ for all θ_i. This is because at any price higher than R_j player i believes with probability one that j is not going to reduce his demand.

Write R for R_i and contrary to the claim of the Lemma, suppose that $p^{n+1} < R$.[5]

For any $b \in (p^{n+1}, R)$, player j could reduce demand either at prices in the interval $[b, R]$ or at prices higher than R (including never reducing demand). Let ζ_θ^b be the probability that player j reduces his demand at prices higher than R, conditional on price b and player j observing θ_j. That is

$$\zeta_\theta^b = \frac{1-\Pi_j(R|\theta_j)}{1-\Pi_j(b|\theta_j)}$$

[5] Assume that $\tilde{R}_j(\theta_j) \geq R$ and $\tilde{R}_i(\theta_i) = R$ for all $\theta \in \Theta$. This is without loss of generality. If there is θ_i' such that say $\tilde{R}_i\left(\theta_i'\right) < R$, then only consider prices greater than $\tilde{R}_i\left(\theta_i'\right)$ in the proof below and only these signals lead player i to reduce her demand after $\tilde{R}_i\left(\theta_i'\right)$. Similarly for signal of player j.

Let $\zeta^b = \min_\theta \zeta^b_\theta$ and let $\left(w^b_{i\theta}, w^b_{j\theta}\right)$ be the expected payoff profile at b conditional on player j reducing his demand at prices less than R and conditional on signal profile being θ. Let $q^b_i(\theta_j|\theta_i)$ to be the posterior probability of θ_j conditional on b and θ_i.

1. It must be that for all θ_i

$$R < \lim_{b \to R} \sum_{\theta_j} q^b_i(\theta_j|\theta_i)\, v_\theta \tag{3}$$

   Clearly, the expected value of the unit at price R cannot be strictly lower than R, because otherwise normal type of both players would have reduced her demand strictly before R, contradicting its definition. Thus, contrary to the statement, assume that there is θ_i such that

$$R = \lim_{b \to R} \sum_{\theta_j} q^b_i(\theta_j|\theta_i)\, v_\theta \tag{4}$$

   Player i of signal θ_i compares the payoff from exiting at at price b and at a later price $b^\# \in (b, R)$. The minimal payoff that this player gets from exiting at b is

$$\sum_{\theta_j} q^b_i(\theta_j|\theta_i)\,(v_\theta - b)\, y$$

   The maximal payoff if this player reduces her demand at $b^\#$ is no higher than

$$\sum_{\theta_j} q^b_i(\theta_j|\theta_i)\left[(v_\theta - b)\, x \left(\Pi_j\left(b^\#|\theta_j\right) - \Pi_j(b|\theta_j)\right) + \left(1 - \Pi_j\left(b^\#|\theta_j\right)\right)\left(v_\theta - b^\#\right) y\right]$$

   which assumes that if j exits at a price in the interval $\left[b, b^\#\right)$, then the quantity x demanded by player i is fully satisfied at the lowest possible price b. The former cannot exceed the latter, because then player i of signal θ_i would reduce her demand before price reaches R. Hence

$$\sum_{\theta_j} q_i^b(\theta_j|\theta_i)\left[(v_\theta - b)\, x\left(\Pi_j\left(b^{\#}|\theta_j\right) - \Pi_j(b|\theta_j)\right) + \left(1 - \Pi_j\left(b^{\#}|\theta_j\right)\right)\left(v_\theta - b^{\#}\right) y\right]$$
$$\geq \sum_{\theta_j} q_i^b(\theta_j|\theta_i)(v_\theta - b)\, y$$

or

$$\sum_{\theta_j'} q_i^b\left(\theta_j'|\theta_i\right)\left(1 - \Pi_j\left(b^{\#}|\theta_j'\right)\right)\sum_{\theta_j} q_i^{b^{\#}}(\theta_j|\theta_i)\left(v_\theta - b^{\#}\right) y$$
$$\geq \sum_{\theta_j} q_i^b(\theta_j|\theta_i)(v_\theta - b)\left[y - x\left(\Pi_j\left(b^{\#}|\theta_j\right) - \Pi_j(b|\theta_j)\right)\right]$$

Now consider this inequality in the limit as $b^{\#}$ goes to R first and then b goes to R. Let $b^{\#} \to R$. The left-hand side converges to zero, by (4). Hence

$$0 \geq \sum_{\theta_j} q_i^b(\theta_j|\theta_i)(v_\theta - b)\left[y - x\left(\lim_{b^{\#}\to R}\Pi_j\left(b^{\#}|\theta_j\right) - \Pi_j(b|\theta_j)\right)\right]$$

Consider the square bracket. Note that for every $\varepsilon > 0$, there is a price b^* strictly less than R such that for all $b > b^*$ and for all θ_j we have $\varepsilon > \lim_{b^{\#}\to R}\Pi_j\left(b^{\#}|\theta_j\right) - \Pi_j(b|\theta_j)$. Replace the square bracket with the term $y - x\varepsilon$, which is strictly lower. Choose $\varepsilon < \frac{y}{x}$, so that $y - x\varepsilon$ is strictly positive. Then we have that for every $b \in (b^*, R)$

$$0 > \sum_{\theta_j} q_i^b(\theta_j|\theta_i)(v_\theta - b)$$

This is a contradiction.

2. The bound ζ^b converges to one: $\lim_{b\to R}\zeta^b = 1$.

   Note that since $\Pi_j(b|\theta_j)$ is non-decreasing, ζ_θ^b is non-decreasing and bounded above, so it is converging. Suppose that there exists θ such that $\lim_{b\to R}\zeta_\theta^b < 1$; that is, player i of signal θ_i thinks that there is a mass point of players j who reduce demand at R. Then there is an $\epsilon > 0$ such that this θ_i reduces his demand at prices less than $R - \epsilon$. (Otherwise player i of signal θ_i, who is supposed to reduce demand at a price arbitrarily close to R, would benefit from delaying this reduction to a price just after R; the gain is bounded away from zero by (3), but the cost of delay is arbitrarily close to zero). This contradicts the definition of R.

3. Total payoff conditional on player j reducing his demand at some price higher than b cannot exceed the total surplus available at price b. Namely, for all θ
$$w_{i\theta}^b + w_{j\theta}^b \leq (v_\theta - b)(x + y) \tag{5}$$

4. With positive probability player i maintains his high demand for all prices less than R; this expected payoff has to be at least as high as the expected payoff from reducing the demand immediately to y at a current price b. Namely, for all θ_i
$$\sum_{\theta_j} q_i^b(\theta_j|\theta_i)\left(\zeta_\theta^b(v_\theta - R)y + \left(1 - \zeta_\theta^b\right)w_{i\theta}^b\right) \geq \sum_{\theta_j} q_i^b(\theta_j|\theta_i)(v_\theta - b)y$$
Replace $w_{i\theta}^b$ by its upper bound implied by inequality (5). Multiply by the denominator of $q_i^\theta(\theta_j|\theta_i)$ and sum over all θ_i, to obtain one equation (where q_θ^b is the probability of θ, given b)
$$\sum_\theta q_\theta^b\left(\zeta_\theta^b(v_\theta - R)y + \left(1 - \zeta_\theta^b\right)\left((v_\theta - b)(x + y) - w_{j\theta}^b\right)\right) \geq \sum_\theta q_\theta^b(v_\theta - b)y \tag{6}$$
Define Θ_L^b to be the set of θ for which
$$(v_\theta - R)y \leq (v_\theta - b)(x + y) - w_{j\theta}^b$$
and let Θ_U^b be its complement. Let $\lambda^b = \sum_{\Theta_U^b} q_\theta^b$.
Write the sums in (6) separately for Θ_L^b and Θ_U^b. Note that for all $\theta \in \Theta_L^b$, one can replace ζ_θ^b with its lower bound ζ^b and the inequality

will hold. Likewise, for all $\theta \in \Theta_U^b$, one can replace ζ_θ^b with its upper bound 1.

$$\sum_{\Theta_L^b} q_\theta^b \left(\zeta^b (v_\theta - R)\, y + \left(1 - \zeta^b\right) \left((v_\theta - b)(x + y) - w_{j\theta}^b\right)\right) \\ + \sum_{\Theta_U^b} q_\theta^b (v_\theta - R)\, y \\ \geq \sum_{\Theta_L^b} q_\theta^b (v_\theta - b)\, y + \sum_{\Theta_U^b} q_\theta^b (v_\theta - b)\, y$$

which can be written as

$$\sum_{\Theta_L^b} q_\theta^b \left(-\zeta^b (R - b)\, y + \left(1 - \zeta^b\right)(v_\theta - b)\, x\right) - \left(1 - \zeta^b\right) \sum_{\Theta_L^b} q_\theta^b w_{j\theta}^b \\ \geq (R - b)\, y \left(1 - \lambda^b\right) \quad (7)$$

Notice that for any b, it must be that $\lambda^b > 0$, or in other words ζ^b occurs with positive probability. If not, then inequality (6) would be violated.

5. With positive probability player j reduces his demand at prices below R. The payoff from this cannot be lower than the payoff from maintaining the demand for x units. Namely, for all θ_j

$$\sum_{\theta_i} q_j^b (\theta_i | \theta_j)\, w_{j\theta}^b \geq \sum_{\theta_i} q_j^b (\theta_i | \theta_j)(v_\theta - R)\, x$$

Multiply this inequality by the denominator of $q_j^b (\theta_i | \theta_j)$ and sum over all θ_j. Then write the sums separately for Θ_L^b and Θ_U^b. Note that $w_{j\theta}^b \leq (v_\theta - b)\, x$; use this observation to replace $w_{j\theta}^b$ with its upper bound in the summation over Θ_U^b to obtain

$$\sum_{\Theta_L^b} q_\theta^b w_{j\theta}^b \geq \sum_{\Theta_L^b} q_\theta^b (v_\theta - R)\, x - (R - b)\, x \left(1 - \lambda^b\right) \quad (8)$$

6. Replace $\sum_{\Theta_L^b} q_\theta^b w_{j\theta}^b$ in inequality (7) with its lower bound in inequality (8)

$$\sum_{\Theta_L^b} q_\theta^b \left(-\zeta^b (R-b)\, y + \left(1-\zeta^b\right)(v_\theta - b)\, x\right)$$
$$- \left(1-\zeta^b\right)\left(\sum_{\Theta_L^b} q_\theta^b (v_\theta - R)\, x - (R-b)\, x \left(1-\lambda^b\right)\right)$$
$$\geq (R-b)\, y \left(1-\lambda^b\right)$$

Group all terms involving summation over Θ_L^b, divide by $R-b>0$ and conclude that ζ^b is bounded away from 1 by a term involving only x and y

$$\zeta^b \leq \frac{x+y\lambda^b - y}{x+y\lambda^b} \leq \frac{x}{x+y} < 1$$

The implication of part 6 contradicts point 2. Thus $R = p^{n+1}$. Since $p^{n+1} < E_{\theta_i|\theta_j,h^{n+1}} v_\theta$, it is impossible that player j reduces his demand at p^{n+1}, so it must be player i. ■

The hypothesis of this inductive step is trivially true for excess demand $n=0$. The result follows.

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